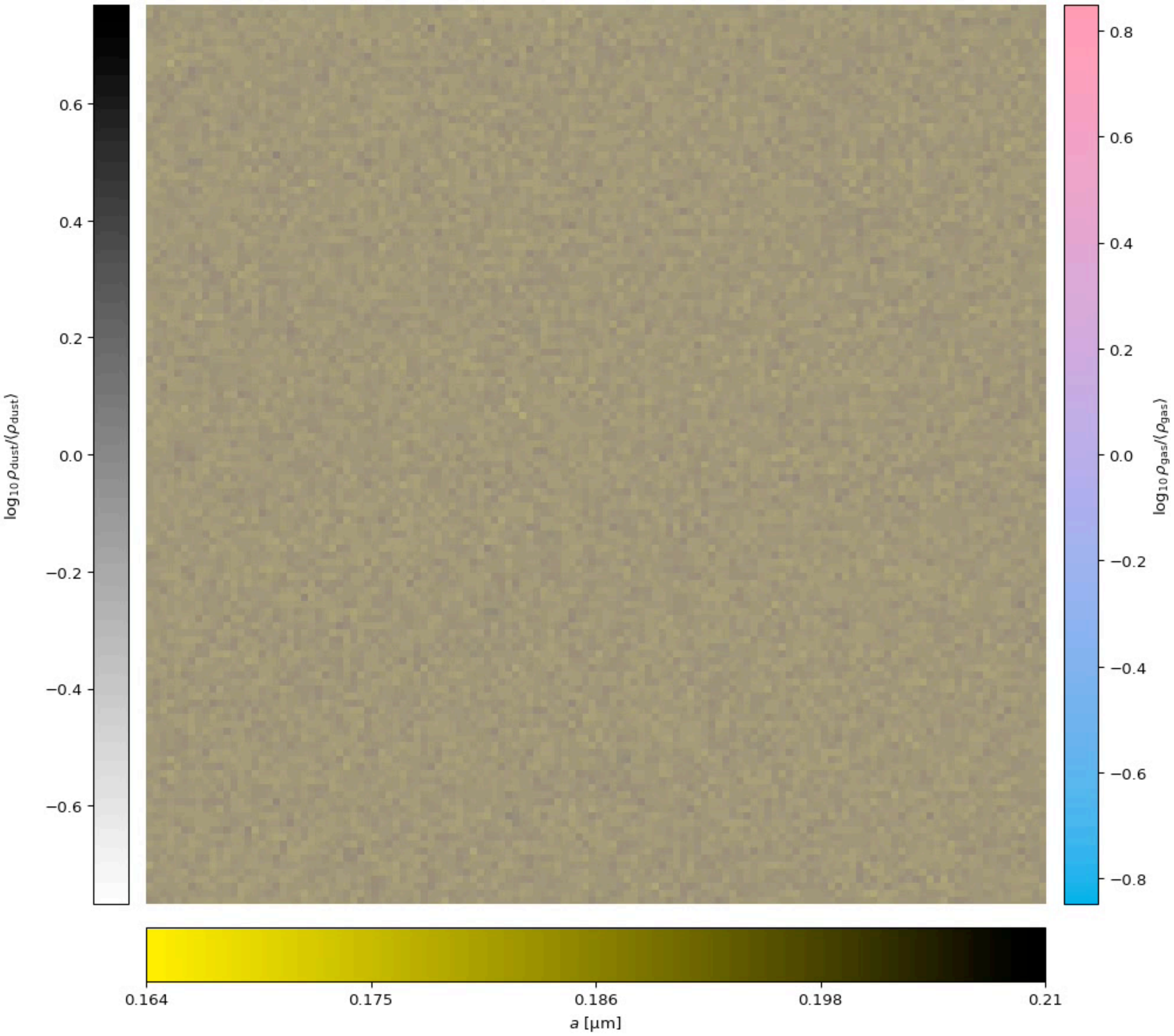
FROM CHARGED PARTICLES TO ITÔ TRACERS: GLUING LAGRANGIAN PARTICLES TO AN EULERIAN FLUID

BY ERIC R. MOSELEY (KIPAC FELLOW @ STANFORD/SLAC)

COLLABORATORS: ROMAIN TEYSSIER, TOM ABEL

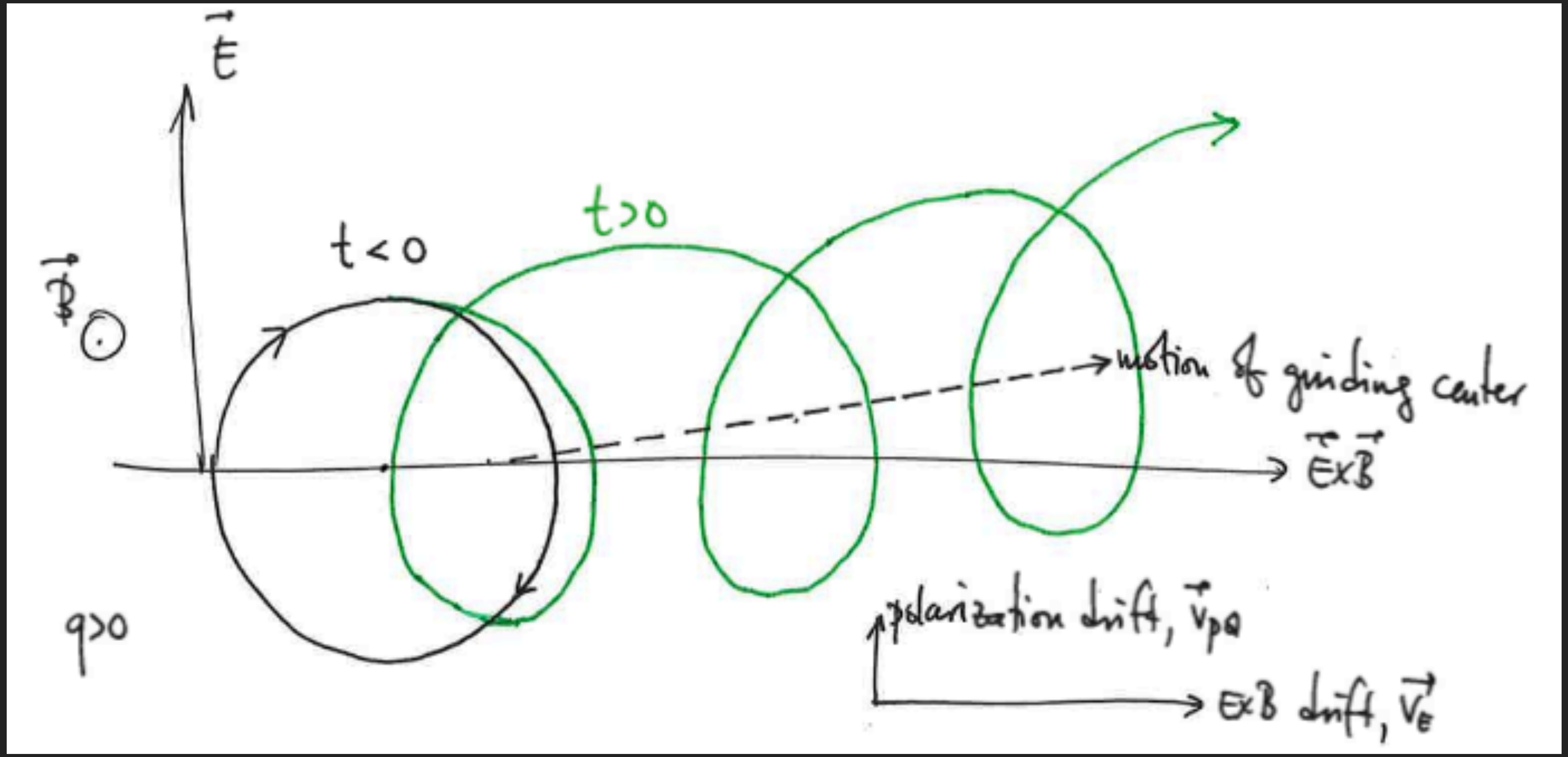


Moseley et al., in prep.

Preliminary!

GUIDING CENTER OF A CHARGED DUST GRAIN IN MAGNETIZED GAS

- A typical grain is *charged*
  - $t_{\rm stop}/t_{\rm Larmor} \gg 1$
  - Possibly even up to $\gtrsim 10^6$
- This is *restrictive.*
- Enter *guiding center theory*
  - Grains are **glued to the gas** transverse to the B-field *(+ small drift corrections)*
  - *Ballistic* along the B-field



(From Prof. Matt Kunz's Plasma Astro lecture notes)

There's just one problem…

HOW DO YOU MAKE A PARTICLE FOLLOW THE GAS???

Itô tracers: continuous-trajectory Lagrangian particles for Eulerian hydrodynamics

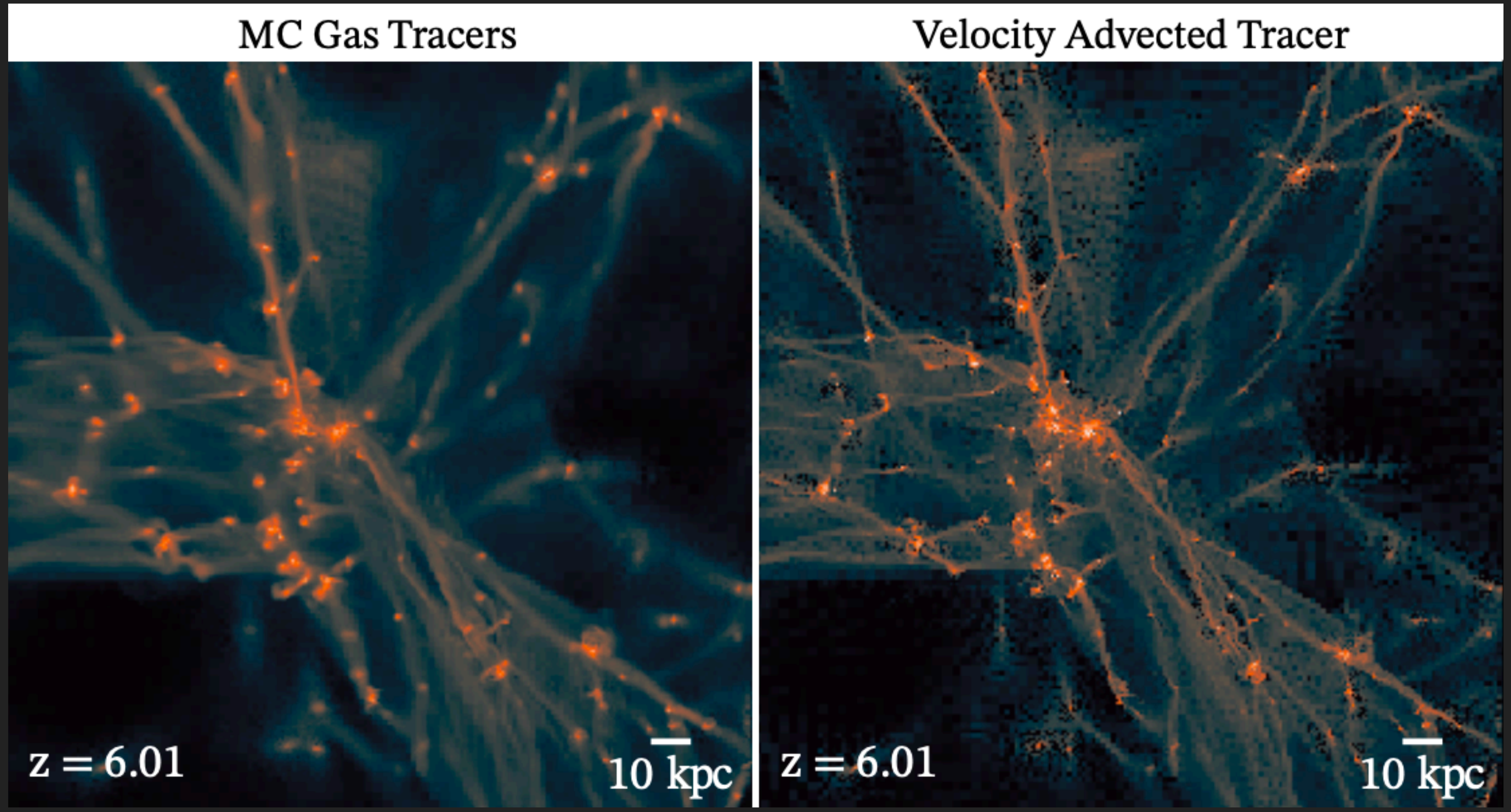
Eric R. Moseley[1,2,3]★, R. Teyssier[4], Tom Abel[1,2,3]

[1] *Kavli Institute for Particle Astrophysics and Cosmology, Stanford University, 452 Lomita Mall, Stanford, CA 94305, USA*
[2] *Department of Physics, Stanford University, 382 Via Pueblo Mall, Stanford, CA 94305, USA*
[3] *SLAC National Accelerator Laboratory, 2575 Sand Hill Road, Menlo Park, CA 94025, USA*
[4] *Department of Astrophysical Sciences, Princeton University, Princeton, NJ 08540, USA*

OUT TOMORROW AT 9 AM!

LAGRANGIAN TRACERS IN EULERIAN CODES

- Eulerian codes are more robust/accurate than Lagrangian for shock capturing, magnetic field studies (CT: $\nabla \cdot B = 0$ to machine precision)
- But *Eulerian* codes do not, by default store *Lagrangian* information
  - Anytime we want the history of a fluid parcel, we need something else (otherwise, use truly Lagrangian methods, SPH, MFM, etc.)
- “Classical” or “Velocity” tracers (e.g. Vazza+ 2011)
  - Deterministic
  - Over-concentrate tracers (10x even)
  - Extreme bias
  - Continuous trajectories; clear mapping onto guiding center theory
- “Monte-Carlo” tracers (e.g. Genel+ 2013, Cadiou+ 2019)
  - Stochastic
  - Unbiased
  - Discontinuous trajectories; not obvious how to map onto e.g. guiding center theory



Cadiou+ 2019

FINITE-VOLUME HYDRODYNAMICS DESCRIBES A MARKOV PROCESS!

- The CFL condition: < 100% of a cell's mass may be transferred to a neighboring cell at a given time-step.
- At each time-step, information is progressively **lost** to numerical diffusion
- Critically, ***evolution for the next time-step is only dependent on a full description of the present state***

The MC tracer is just one of infinitely many possible processes that render the correct *Markov transition kernel*

$$P_{i\to j} \equiv \frac{1}{V_i}\int_{V_i} \mathrm{d}^3x \int_{V_j} \mathrm{d}^3x'\, q(x', t+\Delta t \mid x, t) \;=\; \frac{\max(\Delta M_{ij}, 0)}{M_i}.$$

PROPOSITION: SATISFY THIS WITH A STOCHASTIC DIFFERENTIAL EQUATION!

- SDEs are well studied across many domains (e.g. quantitative finance, brownian motion, etc.)
- Large literature detailing complex machinery for both integration and analysis
- SDEs frequently have continuum analogs for which analytic solutions are known
- Provides a flexible ***framework*** into which we can plug in different terms

HOW TO CONSTRUCT THE ITÔ TRACER: EXPANSION OF A MARKOV PROCESS

$$F_{\rm num} = \sum_{n=1}^{\infty} \frac{\partial^{n-1}}{\partial x^{n-1}} (\kappa_n \, \rho) \implies \frac{\partial \rho}{\partial t} = -\sum_{n=1}^{\infty} \frac{\partial^{n}}{\partial x^{n}} (\kappa_n \, \rho) \, .$$

$$\frac{\partial}{\partial t} p(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left(B_n(x) p(x,t) \right) \iff \mathrm{d}X_t = u \, \mathrm{d}t + \Sigma \, \mathrm{d}W_t$$

- Identifying $\rho_{\rm gas} \to p_{\rm particle}$ allows us to identify a corresponding Markov process theoretically
- Match B_n to κ_n: B_n are effectively the nth moments of the Markov process, but *per unit time.* They are *rates*.
- As a computational tool, we may thus use the discrete MC tracer jump process as a way to compute these coefficients up to order n

THE ITÔ-N FAMILY OF TRACERS

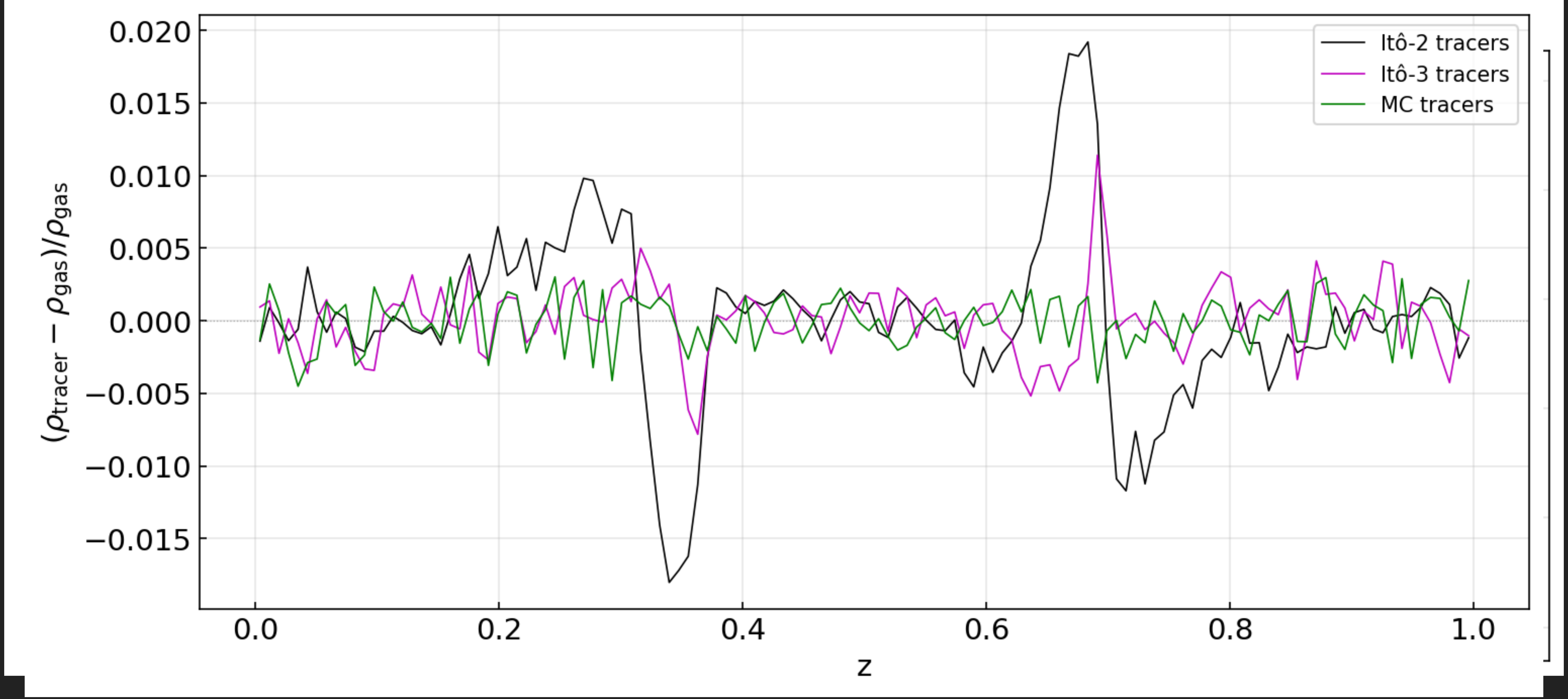
The **Itô-n tracer** is an Itô process matching the first **n** terms in the gas series expansion

$$\mathrm{d}\mathbf{X}_t = \mathbf{u}\,\mathrm{d}t + \Sigma \cdot \mathrm{d}\mathbf{W}_t$$

- $\mathbf{u} = B_1$ is the deterministic *advection velocity*
- $\mathbf{\Sigma} = \sqrt{B_2}$ is a tensor representing a diffusion process
- $\mathbf{W}_t$ is a stochastic process called a **Lévy process**, continuous in time, non-differentiable
- The third moment of $\mathbf{W}_t$ may be assigned to B_3; Generally, the nth moment of $\mathbf{W}_t$ may be assigned to B_n
- ***Without stochastic terms, we can only match the hydro advection equation up to order 1***

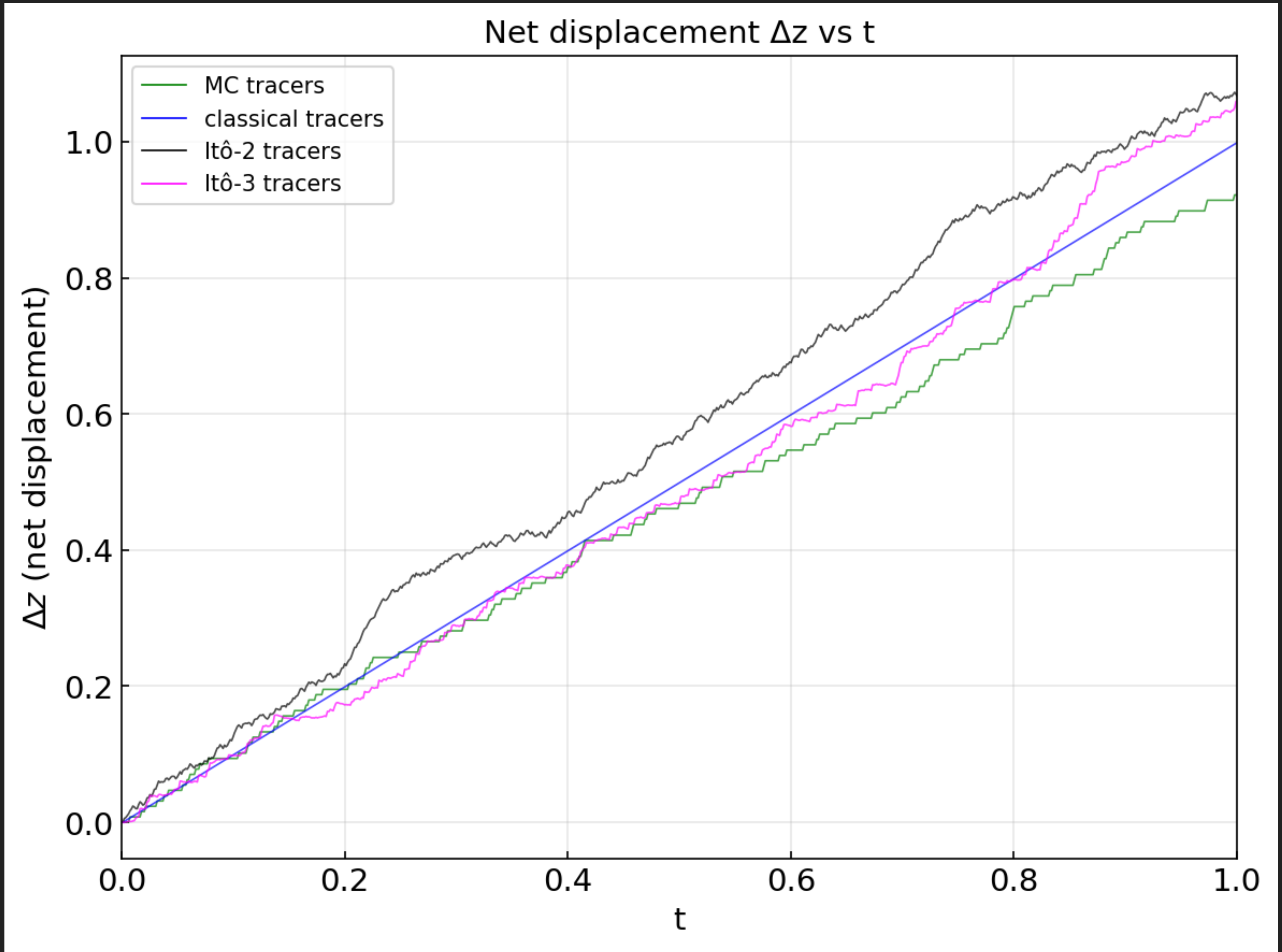
SQUARE PULSE ADVECTION TEST

Moseley, Teyssier, & Abel 2026



- Isothermal, isobaric, single advection period with periodic BCs; N_x = 128, 16–32 particles/cell.
- Itô-n is orders of magnitude better than classical because it has diffusion, *not* because of modified advection

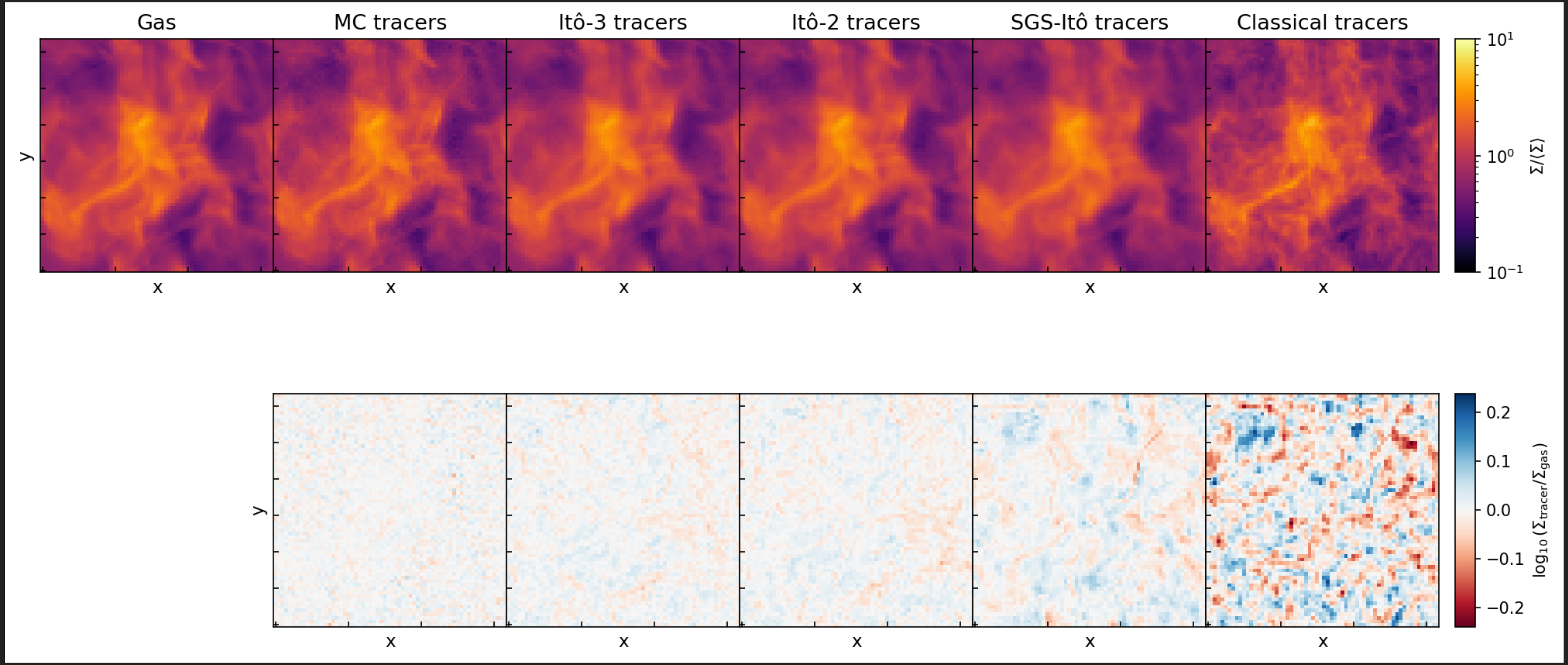
SQUARE PULSE ADVECTION TEST: SAMPLE TRAJECTORIES



Moseley, Teyssier, & Abel 2026

ISOTHERMAL DECAYING TURBULENCE

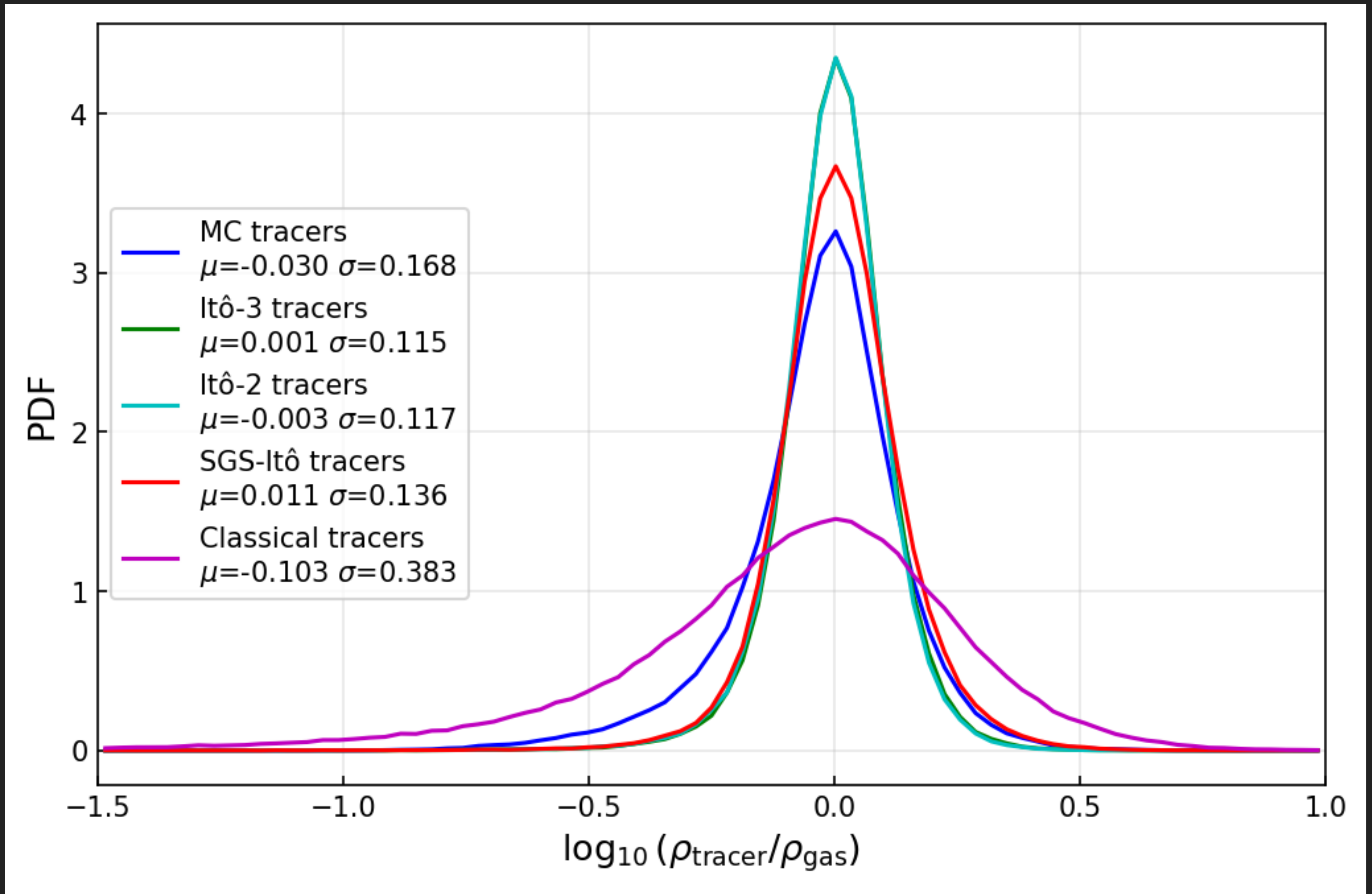
Moseley, Teyssier, & Abel 2026

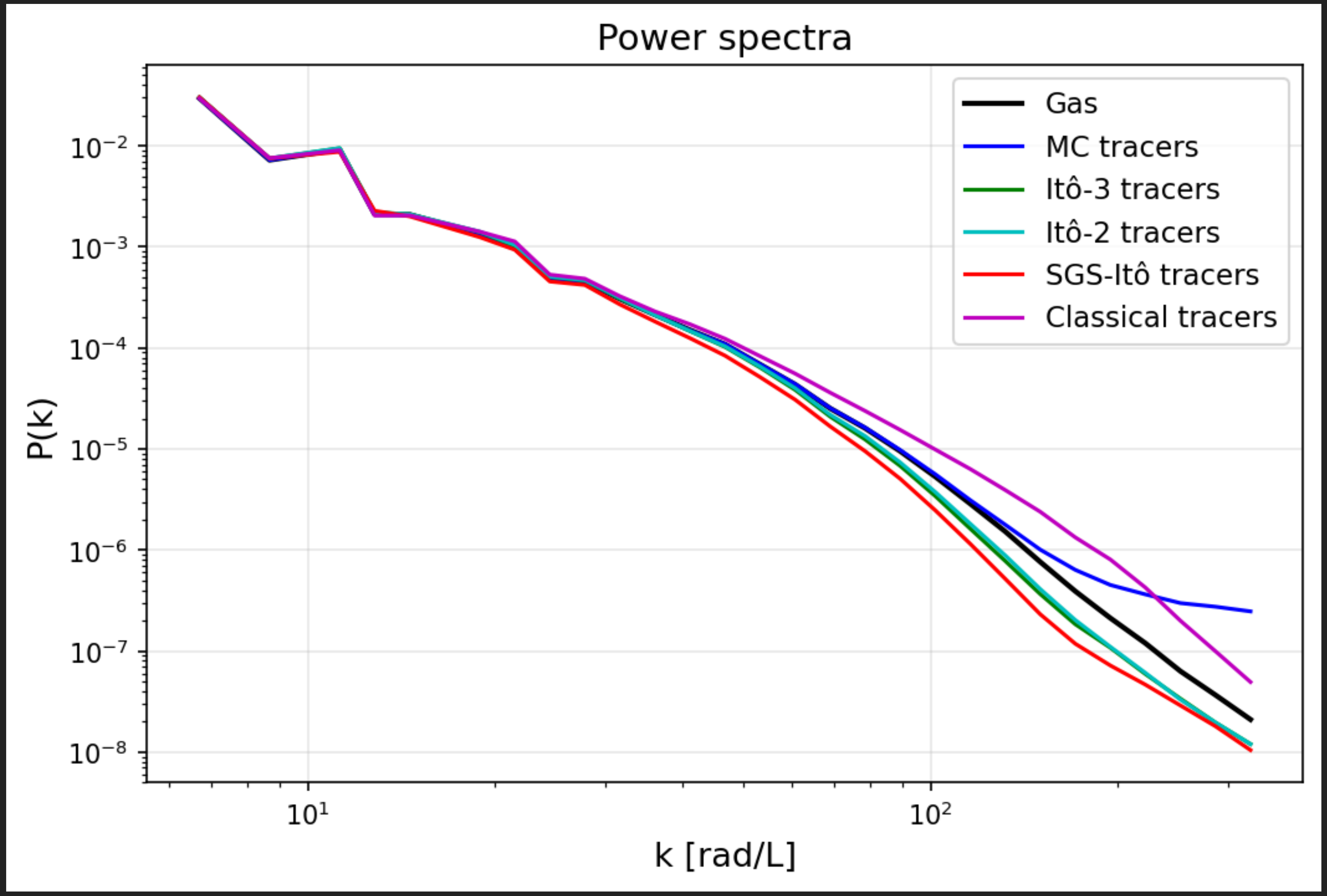


- Isothermal, 3 turnover times; N_x = 64, 16 particles/cell.
- Clear hierarchy in bias and agreement

ISOTHERMAL DECAYING TURBULENCE

Moseley, Teyssier, & Abel 2026



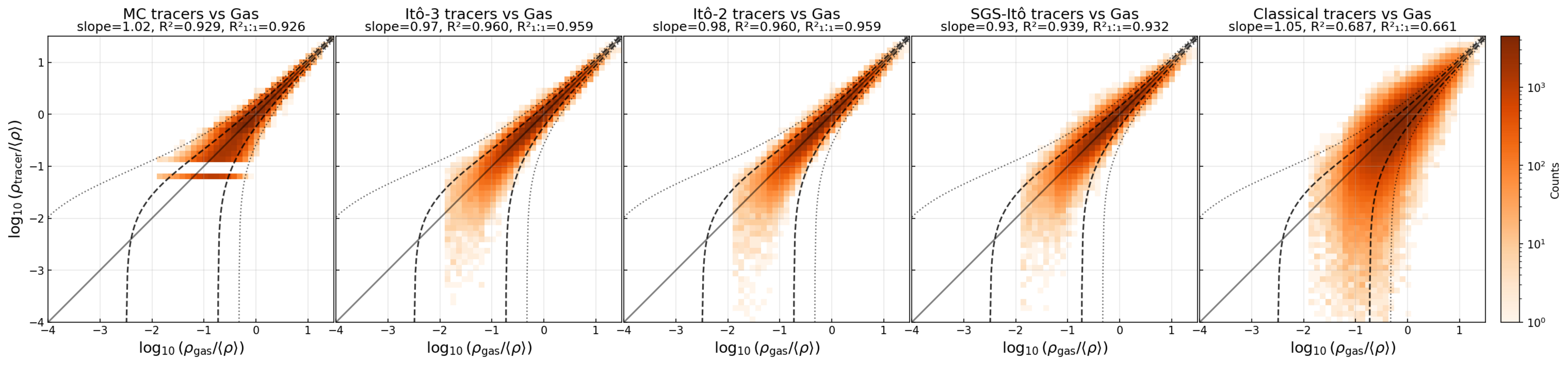


- Interestingly, the Itô-n tracers reduce the width of the log-density-ratio PDF
- Itô-n also exhibits less small-scale power

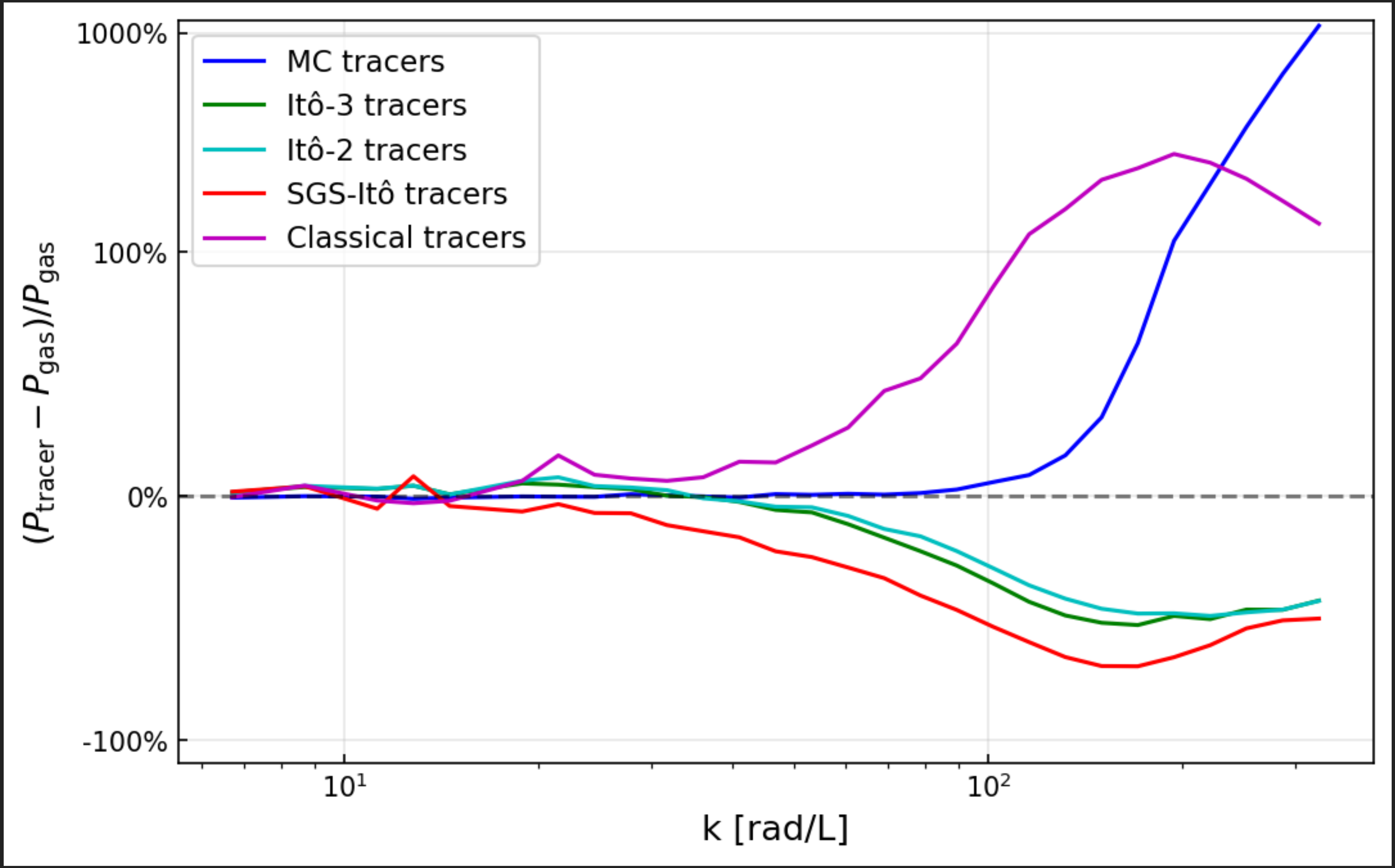
SUMMARY & FUTURE DIRECTIONS

- For many reasons, one may want particles that perfectly follow the gas (perhaps even only for the directions transverse to the **B** field, in the case of charged particles)
- We introduced the **Itô tracer particle**
  - Continuous trajectories obeying a Stochastic Differential Equation (SDE) whose first n displacement moments match those implied by the hydro solver's implied *Markov transition kernel*
  - *Without stochastic terms,* constant-mass, constant-volume Lagrangian particles are <u>fundamentally incapable</u> of matching the gas evolution
  - Like the gas, tracer particles must be able to dissipate small-scale power $\Longrightarrow$ stochastic diffusion
- Itô tracers improve upon MC tracers in many measures, while also providing a *framework upon which other physics can be built*: e.g., **guiding center dust**

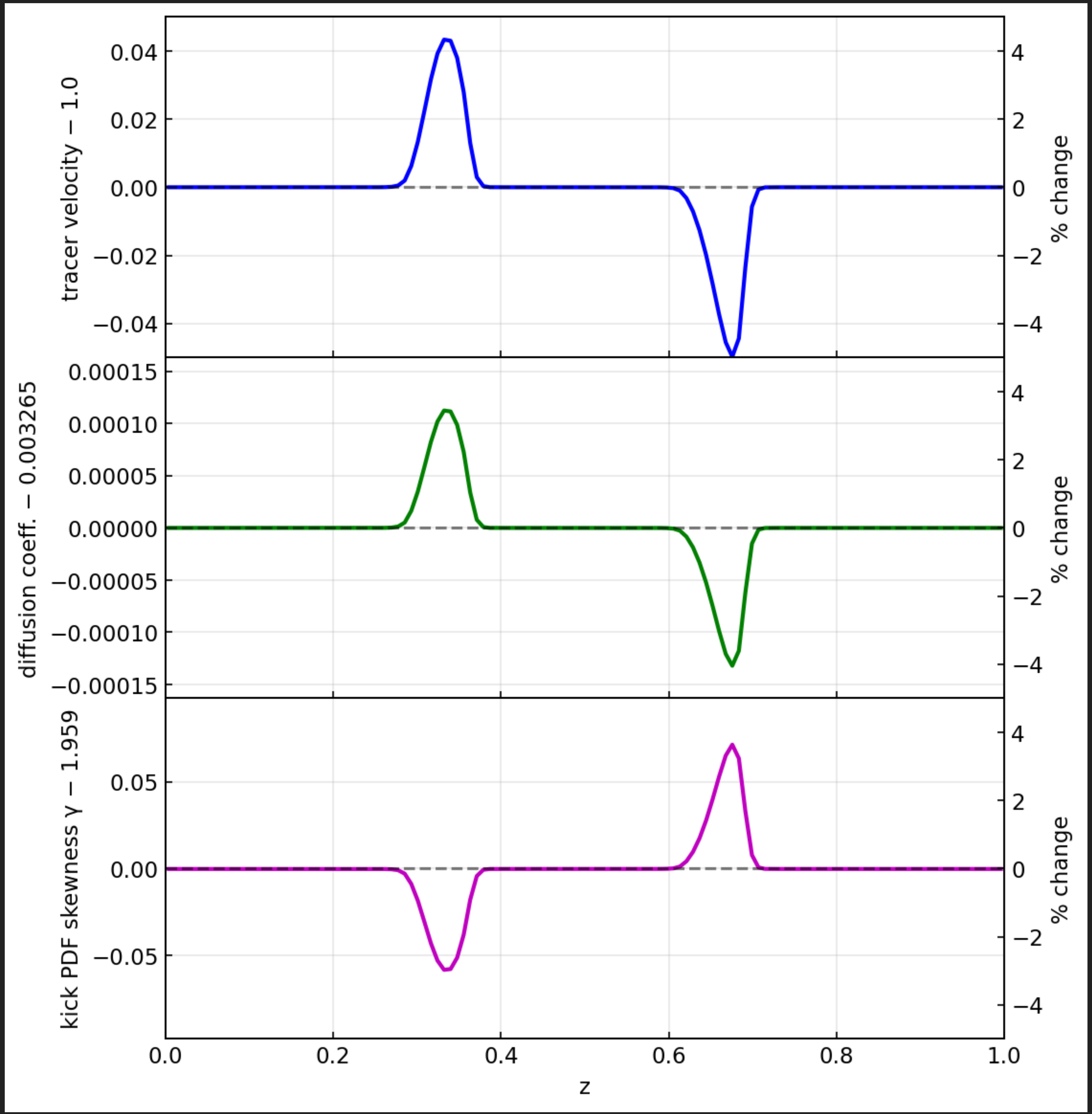
BACKUP: JOINT DENSITY HISTOGRAMS



BACKUP: POWER-SPECTRUM RELATIVE ERROR



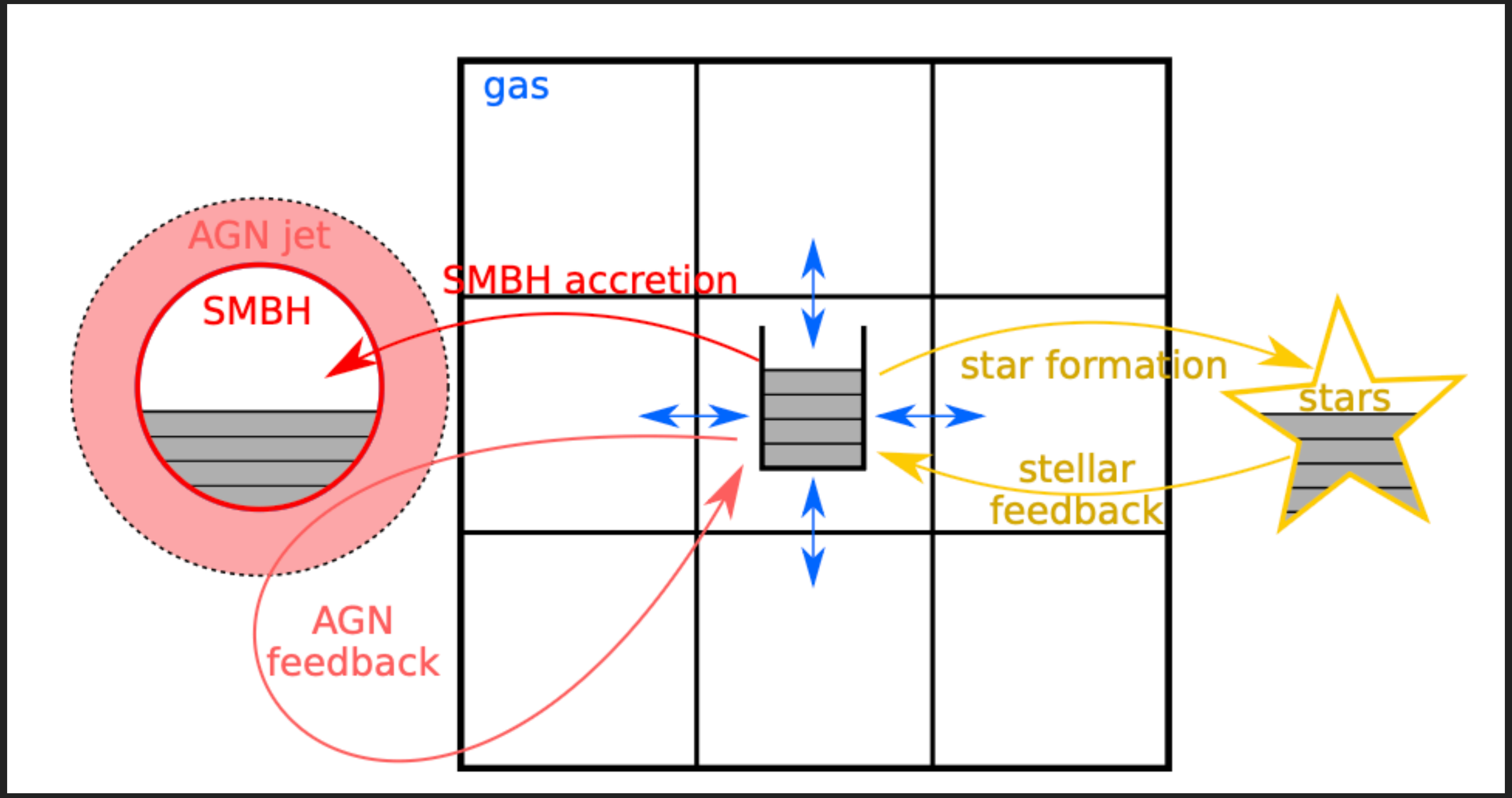
BACKUP: LOCAL DRIFT, DIFFUSION, SKEWNESS ACROSS THE PULSE



MONTE-CARLO (MC) TRACERS: A DISCRETE-JUMP MARKOV PROCESS

- Genel et al. 2013; Cadiou et al. 2019: :
  - Attach the tracer to its *host cell*. Each step, jump to neighbor j with probability $P_{i \to j} \equiv \dfrac{\Delta M_{ij}}{M_i}$
  - In the $N_{\text{tracer}} \to \infty$ limit, converges to the gas distribution with Poisson errors
- Trajectories are *discontinuous in time*; they do not admit differential equation analysis
- Techniques developed for *stochastic differential equations* (SDEs) are inaccessible, and “paths" are not rigorously defined
- Variance reduction, higher order integrals, formal path-level analysis are thus unavailable
- Data-dependent cell exchange is awkward on GPUs
- Advection is *stochastic*, even though it has no reason to be (as we’ll see)

Schematic representing the MC tracer method



Cadiou+ 2019

HOW TO CONSTRUCT THE ITÔ PROCESS: THE KRAMERS-MOYAL EXPANSION

Gas satisfies:

$$B_1^{\mathrm{MC}} \equiv \frac{\langle \Delta X \rangle}{\Delta t}$$

Markov particles satisfy:

$$B_n^{\mathrm{MC}} \equiv \frac{\mathbb{E}\left[\left(\Delta X - \langle \Delta X \rangle\right)^n \,\middle|\, X_t = x_i\right]}{\Delta t}$$

- “Modified equation analysis”
- Identifying $\rho_{\text{gas}} \to p_{\text{particle}}$ allows us to identify a corresponding Markov process theoretically
- Match B_n to κ_n
- However, *the* limit definition is only well defined for a continuous process; ill-defined for the MC tracer.
- As a computational tool, we may thus use the discrete MC tracer jump process as a way to compute these coefficients up to order n: